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IMPROVED LOWER BOUNDS FOR THE UNIFORM RADIUS OF SPATIAL ANALYTICITY OF THE MODIFIED CAMASSA-HOLM EQUATION

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ABSTRACT. We show that the uniform radius of spatial analyticity, $\sigma(t)$, of the solutions at time t for the modified Camassa-Holm equation is bounded below by $c|t|^{-\frac{1}{2\gamma}}$ for large t, where γ is a value in the interval $(-0, 1]$, provided the initial data is analytic with a fixed radius σ_0. To establish this lower bound, we use the standard contraction mapping principle, an approximate conservation law in the modified Gevrey space $H^{\sigma,1}$, linear estimates, a Strichartz estimate, the Transference principle, and Sobolev embedding. This result enhances the findings of Himonas and Petronilho, as well as Getachew.

1. Introduction

We consider the modified Camassa-Holm (mCH) equation

$$\partial_t u + (-1)^{1+j}\partial_x^{2j+1}u + u\partial_x u + \partial_x(1-\partial_x^2)^{-1}\left(u^2 + \frac{1}{2}(\partial_x u)^2\right) = 0, \tag{1.1}$$

where the unknown function $u(x,t)$ is real-valued and j is any integer such that $6 \leqslant j \leqslant 10$. This type of equation originally introduced in [8, 9] by Himonas and Misiołek as a dispersive regularization of the well-known Camassa-Holm (CH) equation, which is an integrable equation arising in the water wave theory.

We compliment (1.1) with initial data

$$u(x,0) = u_0(x). \tag{1.2}$$

The energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}}\left(u^2 + (\partial_x u)^2\right)dx \tag{1.3}$$

is conserved by the flow of (1.1). That is

$$E(t) = E(0) \quad \forall t. \tag{1.4}$$

Well-posedness of (1.1) in the Sobolev space has been studied intensively by various scholars. In particular, Li *et al.* [17] proved that (1.1) is locally well-posed for initial data in $H^s(\mathbb{R})$ with $s > -j + \frac{5}{4}$, where $j \geqslant 1$ is any integer. Moreover, the authors proved (1.1) is globally well-posed for initial data in $H^1(\mathbb{R})$. Later, Li *et al.* [16] proved sharp well-posedness for the small initial data in $H^{-j+\frac{5}{4}}(\mathbb{R})$

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and ill-posedness for the initial data in homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R})$ with $s < -j + \frac{5}{4}$ to (1.1), where $j \geqslant 2$ be any integer.

The idea of spatial analyticity radius was introduced in [14] by Kato and Masuda. Since then, numerous authors have focused on establishing an algebraic decay rate of order $t^{-\alpha}$ (for some $\alpha \geqslant 1$) for the radius of spatial analyticity in various nonlinear dispersive partial differential equations (PDEs) (see for example, [1,5,6,18] and reference therein).

Coming back to (1.1), Himonas and Petronilho [11] studied the property of the spatial analyticity of the solution $u(x,t)$, given that the initial data $u_0(x)$ is real-analytic with uniform radius of analyticity σ_0, so there is a holomorphic extension to a complex strip S_{σ_0} of width $2\sigma_0$. In fact, the authors studied the initial value problem (IVP) associated with the modified CH equation (1.1) and obtained a lower bound on the radius of spatial analyticity

$$\sigma(T) = \min\{\sigma_0, cT^{-\alpha}\},$$

where $\alpha = \frac{4}{3} + \epsilon$, for sufficiently small $\epsilon > 0$, if $j = 1$ and $\alpha = 1$ if $j \geqslant 2$ is any integer.

For $j \geqslant 9$, the result of Himonas and Petronilho is improved in the paper [12]. The author obtained a decay rate of order $c|t|^{-\frac{4}{5}}$ when $j = 9$, $c|t|^{-\frac{4}{7}}$ when $j = 10$, and $c|t|^{-\frac{1}{2}}$ when $j \geqslant 11$.

This paper aims to improve the results obtained in [11,12]. To achieve this, we first introduce crucial function spaces for the study of spatial analyticity.

To study the spatial analyticity radius for solutions of a large class of dispersive partial differential equations (PDEs), we introduce the Gevrey space of analytic functions, denoted by $G^{\sigma,s} := G^{\sigma,s}(\mathbb{R})$ for $s \geqslant 0$ and $\sigma > 0$. This space is defined via the norm

$$\|f\|_{G^{\sigma,s}} = \left\| e^{\sigma|\xi|} \langle\xi\rangle^s \widehat{f} \right\|_{L^2_\xi(\mathbb{R})},$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ and $\widehat{f}$ is the spatial Fourier transform of f given by

$$\widehat{f}(\xi) := \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

Note that when $\sigma = 0$, the Gevrey spaces $G^{\sigma,s}(\mathbb{R})$ coincide with the Sobolev spaces $H^s := H^s(\mathbb{R})$, which are equipped with the norm

$$\|f\|_{H^s} = \|\langle\xi\rangle^s \hat{f}\|_{L^2_\xi(\mathbb{R})},$$

while for $\sigma > 0$, any function in $G^{\sigma,s}(\mathbb{R})$ has a radius of analyticity of at least σ at each point $x \in \mathbb{R}$. This fact is contained in the Paley–Wiener Theorem. A proof for the case $s = 0$ can be found in [15]; the general case follows from a simple modification.

Theorem 1 (Paley–Wiener Theorem). *Let $\sigma > 0$ and $s \in \mathbb{R}$. Then, any function f is in $G^{\sigma,s}(\mathbb{R})$ if and only if f is the restriction to $\mathbb{R}$ of a function F which is holomorphic in the strip*

$$S_\sigma = \{x + iy \in \mathbb{C} : \ |y| < \sigma\}.$$

Moreover, the function F satisfies the estimates

$$\sup_{|y|<\sigma} \|F(\cdot + iy)\|_{H^s(\mathbb{R})} < \infty.$$

To obtain a better lower bound for the radius of spatial analyticity for solutions to nonlinear PDEs, Dufera *et al.* [4] introduced a new idea to establish a higher order conservation law. For this purpose, the authors introduced a new space called a modified Gevrey space, denoted by $H^{\sigma,s} := H^{\sigma,s}(\mathbb{R})$, which is endowed with the norm

$$\|f\|_{H^{\sigma,s}} = \left\|\cosh(\sigma|\xi|)\langle\xi\rangle^s \widehat{f}\right\|_{L^2_\xi(\mathbb{R})}.$$

This space is obtained from the Gevrey space $G^{\sigma,s}(\mathbb{R})$ by replacing the exponential weight $e^{\sigma|\xi|}$ with the hyperbolic weight $\cosh(\sigma|\xi|)$.

The two weights are equivalent in the sense that

$$\frac{1}{2}e^{\sigma|\xi|} \leqslant \cosh(\sigma|\xi|) \leqslant e^{\sigma|\xi|}. \tag{1.5}$$

As a consequence of (1.5), the norms $\|f\|_{H^{\sigma,s}}$ and $\|f\|_{G^{\sigma,s}}$ are equivalent; i.e.,

$$\|f\|_{H^{\sigma,s}} \sim \|f\|_{G^{\sigma,s}}, \tag{1.6}$$

and hence the statement of Paley-Wiener Theorem still holds for functions in $H^{\sigma,s}(\mathbb{R})$.

The method of Gevrey approximate conservation laws yields a decay rate of order $t^{-1/\rho}$ for some $0 < \rho \leqslant 1$ on the radius of spatial analyticity of solutions to a number of nonlinear dispersive and wave equations (see, e.g, [10,22–26] and the references therein). This decay rate is obtained based on the simple estimate

$$1 - e^{-\sigma|\xi|} \leqslant (\sigma|\xi|)^\rho.$$

In an attempt to improve the decay rate, the use of approximate conservation laws in the modified Gevrey space can yield a decay rate of order $t^{-1/(2\rho)}$ for some $0 < \rho \leqslant 1$ (see, e.g., [13,20,21,27] and references therein). This decay rate is obtained using the inequality

$$\cosh(\sigma|\xi|) - 1 \leqslant (\sigma|\xi|)^{2\rho}\cosh(\sigma|\xi|),\ 0 \leqslant \rho \leqslant 1.$$

This inequality follows from an interpolation between the cases where $\cosh r - 1 \leqslant \cosh r$ and $\cosh r - 1 \leqslant r^2 \cosh r$ for $r \in \mathbb{R}$.

Observe that the modified Gevrey space satisfy the following embedding property:

$$H^{\sigma,s}(\mathbb{R}) \subset H^{\sigma',s'}(\mathbb{R}) \quad \text{for all } 0 \leqslant \sigma' < \sigma \text{ and } s \leqslant s' \in \mathbb{R}. \tag{1.7}$$

In particular, we have the embedding

$$H^{\sigma,s}(\mathbb{R}) \subset H^{s'}(\mathbb{R}) \quad \text{for all} 0 < \sigma \text{ and } s \leqslant s' \in \mathbb{R}.$$

As a consequence of this embedding property and the existing well-posedness theory in $H^1(\mathbb{R})$, we conclude that the IVP (1.1)–(1.2) has a unique and global-in-time solution, given initial data $u_0 \in H^{\sigma_0,1}(\mathbb{R})$ for all $\sigma_0 > 0$.

Our main result is as follows:

Theorem 2. *Suppose that* u *is the global solution of* (1.1)–(1.2) *with initial data* $u_0 \in H^{\sigma_0,1}$ *for some* $\sigma_0 > 0$. *Then*

$$u(t) \in H^{\sigma(t),1} \quad \forall t, \tag{1.8}$$

with the radius of analyticity $\sigma(t)$ *satisfying the lower bound*

$$\sigma(t) \geqslant c|t|^{-\frac{2}{3}}, \quad \textit{when} \;\; j=6, \tag{1.9}$$

$$\sigma(t) \geqslant c|t|^{-\frac{1}{2}}, \quad \textit{when} \;\; 7 \leqslant j \leqslant 10, \tag{1.10}$$

where $c > 0$ *is a constant depending on* $\|u_0\|_{H^{\sigma_0,1}}$ *and* $6 \leqslant j \leqslant 10$ *is any integer.*

Notation: For $a, b > 0$ in $\mathbb{R}$, we use $a \lesssim b$ if there exists a positive constant C, which may vary from line to line such that $a \leqslant Cb$. Moreover, we use $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. Furthermore, for any $\epsilon > 0$ and any $a \in \mathbb{R}$, we use $a\pm$ to mean $a \pm \epsilon$.

The mCH equation is invariant under the change of variables $(x,t) \to (-x,-t)$, so we can assume that time $t \geqslant 0$. The rest of the paper is organized as follows. In Section 2, we introduce the necessary function spaces and establish various space-time estimates used throughout the paper. We also state the local-in-time well-posedness result in this section. In Section 3, we derive an approximate conservation law in the space $H^{\sigma,1}(\mathbb{R})$ because the H^1-norm is conserved by the flow of (1.1). Finally, in Section 4, we establish the lower bounds given in (1.9) for the uniform radius of spatial analyticity, which completes the proof of Theorem 2.

2. Preliminary concepts

In this section, we discuss the function spaces and space-time estimates used throughout this paper. We also state the local well-posedness result for the initial value problem (IVP) (1.1)–(1.2) in the modified Gevrey space $H^{\sigma,1}$ for $\sigma > 0$.

The Bourgain space $X^{s,b} := X^{s,b}(\mathbb{R}^2)$ associated with (1.1) is defined as the completion of the Schwartz space $\mathscr{S}_{x,t}(\mathbb{R} \times \mathbb{R})$ with respect to the norm

$$\|u\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau - \xi^{2j+1}\rangle^b \widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R})},$$

where $s, b \in \mathbb{R}$ and $\widetilde{u}$ denotes the space-time Fourier transform of u given by

$$\widetilde{u}(\xi,\tau) := \mathcal{F}_{x,t}[u](\xi,\tau) = \int_{\mathbb{R}^2} e^{-i(t\tau + x\xi)} u(x,t)\, dxdt.$$

For any $\sigma > 0, s, b \in \mathbb{R}$, the Gevrey-Bourgain space , denoted by $X^{\sigma,s,b} := X^{\sigma,s,b}(\mathbb{R}^2)$, is defined by the norm

$$\|u\|_{X^{\sigma,s,b}} = \|\langle\xi\rangle^s \langle\tau - \xi^{2j+1}\rangle^b e^{\sigma|\xi|} \widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R})}.$$

This space coincides with the Bourgain space for $\sigma = 0$.

Finally, we will need the restriction of $X^{s,b}$ and $X^{\sigma,s,b}$ to a time slab $(0,T) \times \mathbb{R}$. These restricted spaces, denoted by $X^{s,b}_T$ and $X^{\sigma,s,b}_T$ respectively, are Banach spaces equipped with the norms

$$\|u\|_{X^{s,b}_T} := \inf\{\|v\|_{X^{s,b}} : \; v = u \text{ on } (0,T) \times \mathbb{R}\}$$

$$\text{Resp., } \|u\|_{X^{\sigma,s,b}_T} := \inf\{\|v\|_{X^{\sigma,s,b}} : \; v = u \text{ on } (0,T) \times \mathbb{R}\}.$$

Now, we record some useful estimates that will be used in this paper. To do this, we first consider the IVP for the linear part of the mCH equation

$$\begin{cases} \partial_t u + (-1)^{j+1}\partial_x^{2j+1} u = F(x,t), \\ u(x,0) = u_0(x). \end{cases} \tag{2.1}$$

Using Duhamel's formula, we can write the IVP (2.1) in its integral equation form as

$$u(t) = W_j(t)u_0 - \int_0^t W_j(t-\tau)F(\tau)d\tau, \tag{2.2}$$

where $W_j(t) := e^{(-1)^j t\partial_x^{2j+1}}$ is the solution group. The solution group $W_j(t)$ satisfies the following energy inequalities. For a detailed proof, we refer to [11, Lemma 2.1]

Lemma 2.1. *Let* $\sigma \geqslant 0$ *and* $1/2 < b \leqslant 1$*. Then, for all* $0 < \delta \leqslant 1$*, there is a constant* $C := C(b)$ *such that*

$$\left\| W_j(t)u_0 \right\|_{X_\delta^{\sigma,1,b}} \leqslant C \left\| u_0 \right\|_{H^{\sigma,1}}, \tag{2.3}$$

$$\left\| \int_0^t W_j(t-\tau)F(\tau)\, d\tau \right\|_{X_\delta^{\sigma,1,b}} \leqslant C \left\| F \right\|_{X_\delta^{\sigma,1,b-1}}. \tag{2.4}$$

Moreover, we have the following $L_\delta^q L_x^p$ estimates for $W_j(t)u_0(x)$ [3].

Lemma 2.2 (Strichartz estimate)**.** *Let* $j \geqslant 1$ *be any integer such that* $-1 < \alpha \leqslant \frac{2j-1}{2}$ *and* $0 \leqslant \theta \leqslant 1$*. Then for any* $\delta > 0$*, there exists a constant* $C > 0$ *depending only on* δ, α *and* θ *such that for any* $0 < T \leqslant \delta$ *and* $u_0 \in L_x^2(\mathbb{R})$

$$\left\| D^{\frac{\theta\alpha}{2}} W_j(t)u_0 \right\|_{L_T^q L_x^p} \leqslant C\|u_0\|_{L_x^2},$$

where $p = 2/(1-\theta)$ *and* $q = (4j+2)/\theta(1+\alpha)$*.*

With a simple modification, one can deduce from Lemma 2.2 that

$$\left\| \langle D\rangle^{\frac{\theta\alpha}{2}} S_j(t)u_0 \right\|_{L_T^q L_x^p} \leqslant C\|u_0\|_{L_x^2}. \tag{2.5}$$

Then, by a standard transference principle[1], the estimate from (2.12) implies

$$\left\| \langle D\rangle^{\frac{\theta\alpha}{2}} u \right\|_{L_T^q L_x^p} \leqslant C\|u\|_{X^{0,b}}, \tag{2.6}$$

for all $u \in X^{0,b}$, where $j, \alpha, \theta, \delta, T, p$, and q are as in Lemma 2.2 and $b > \frac{1}{2}$.

In particular, choosing $\theta = 1$ and $2 \leqslant \alpha \leqslant \frac{2j-1}{2}$ yields

$$\left\| \langle D\rangle^{\frac{\alpha}{2}} u \right\|_{L_T^4 L_x^\infty} \leqslant C\|u\|_{X^{0,b}}, \tag{2.7}$$

and choosing $\theta = 0$ yields

$$\|u\|_{L_T^\infty L_x^2} \leqslant C\|u\|_{X^{0,b}}. \tag{2.8}$$

[1] Its proof can be found in [19, Lemma 2.9].

On the other hand, choosing $\theta = 1$ and $\alpha = 2$ associated with $j \geqslant 3$ yields

$$\|\langle D\rangle u\|_{L_T^{\frac{4j+2}{3}} L_x^\infty} \leqslant C\|u\|_{X^{0,b}}. \tag{2.9}$$

For $1/2 < b \leqslant 1$ and $\delta > 0$, we have[2]

$$\|u\|_{X_\delta^{s,b-1}} \leqslant CT^{1-b}\|u\|_{L_\delta^2 H^s}, \tag{2.10}$$

where $C > 0$ is a constant independent of δ.

For the specific case of $j = 6$, the Lemma gives

$$\left\| D^{\frac{\theta\alpha}{2}} W_6(t)u_0 \right\|_{L_T^q L_x^p} \leqslant C\|u_0\|_{L_x^2}, \tag{2.11}$$

where $-1 < \alpha \leqslant \frac{11}{2}$, and θ, δ, T, p, and q are as in Lemma 2.2.

A simple modification of (2.11) yields:

$$\left\| \langle D\rangle^{\frac{\theta\alpha}{2}} W_6(t)u_0 \right\|_{L_T^q L_x^p} \leqslant C\|u_0\|_{L_x^2}. \tag{2.12}$$

Applying a standard transference principle (see [19, Lemma 2.9]) to the estimate in (2.12) gives:

$$\left\| \langle D\rangle^{\frac{\theta\alpha}{2}} u \right\|_{L_T^q L_x^p} \leqslant C\|u\|_{X^{0,b}}, \tag{2.13}$$

for all $u \in X^{0,b}$.

Specifically, choosing $\theta = 1$ and $\alpha = \frac{11}{2}$ leads to:

$$\left\| \langle D\rangle^{\frac{11}{4}} u \right\|_{L_T^4 L_x^\infty} \leqslant C\|u\|_{X^{0,b}}. \tag{2.14}$$

In the same way, choosing $\theta = 1$ and $\alpha = 4$ leads to:

$$\left\| \langle D\rangle^2 u \right\|_{L_T^4 L_x^\infty} \leqslant C\|u\|_{X^{0,b}}. \tag{2.15}$$

On the other hand, choosing $\theta = 0$ yields

$$\|u\|_{L_T^\infty L_x^2} \leqslant C\|u\|_{X^{0,b}}. \tag{2.16}$$

We need the following lemma, whose proof can be found in [4, Lemma 3].

Lemma 2.3. *Let $p \geqslant 1$ be an integer, $\sigma > 0$, and $\xi_m \in \mathbb{R}$ for $m = 1, 2, \cdots, p$ such that $\xi = \sum_{m=1}^{p} \xi_m$. If $K(\sigma\xi) := 1 - \cosh(|\sigma\xi|) \prod_{m=1}^{p} \operatorname{sech}(\sigma|\xi_m|)$, then*

$$|K(\sigma\xi)| \leqslant 2^p \sigma^2 \sum_{m \neq n = 1}^{p} |\xi_m||\xi_n|.$$

We conclude this section with the following local well-posedness result, whose proof can be found in [12, Theorem 2].

Theorem 3 (Local well-posedness)**.** *Let $\sigma > 0$. Then for any $u_0 \in H^{\sigma,1}$ there exists a time $\delta = \delta(\|u_0\|_{H^{\sigma,1}}) > 0$ and a unique solution*

$$u \in C\left([0, \delta]; H^{\sigma,1}\right),$$

[2]It can be found in [19, Lemma 2.11].

of the IVP (1.1) *on* $\mathbb{R} \times [0, \delta]$. *Moreover, the solution depends continuously on the data* u_0 *and the existence time is given by*

$$\delta = c_0 \|u_0\|_{H^{\sigma,1}}^{-a}, \tag{2.17}$$

for some constants $c_0 > 0$ *and* $a > 1$. *In addition, for some* $1/2 < b < 1$, *the solution* u *satisfies the bound*

$$\|u\|_{X_\delta^{\sigma,1,b}} \leqslant C\|u_0\|_{H^{\sigma,1}}. \tag{2.18}$$

3. Approximate conservation law

Let

$$v_\sigma(x,t) := \cosh(\sigma|D|)u(x,t),$$

where u is the local solution to (1.1). Then, $u = \operatorname{sech}(\sigma|D|)v_\sigma$. Note also that $v_\sigma(\cdot, 0) = \cosh(\sigma|D|)u_0(\cdot)$.

Now, define the modified energy associated with the function v_σ as:

$$E_\sigma(t) := \frac{1}{2}\int_{\mathbb{R}} \left(v_\sigma^2 + (\partial_x v_\sigma)^2\right) dx. \tag{3.1}$$

For $\sigma = 0$, we have from (1.3) the conservation

$$E_0(t) = E_0(0), \ \forall t.$$

However, this conservation property fails to hold for $\sigma > 0$. In what follows, we will nevertheless find a growth estimate for $E_\sigma(t)$ and consequently prove that it is an approximate conserved quantity at the H^1 level of Sobolev regularity. This will allow us to extend the local solution to the IVP (1.1)–(1.2) globally in time and obtain a lower bound for the spatial analyticity radius $\sigma(t)$ as $|t| \to \infty$.

For $0 < \tau \leqslant \delta$, and δ as in Theorem 3, we have

$$E_\sigma(\tau) = E_{\sigma_0}(0) + R_\sigma(\tau), \tag{3.2}$$

where

$$R_\sigma(\tau) := \int_0^\tau \int_{\mathbb{R}} v_\sigma \cdot N(v_\sigma)\, dx dt, \tag{3.3}$$

with $N(v_\sigma)$ as in (3.6).

Proof of (3.2). Because of integration by parts[3], (1.1) can be rewritten as

$$\partial_t u - \partial_t\partial_x^2 u + (-1)^{j+1}\partial_x^{2j+1}u + (-1)^{j+2}\partial_x^{2j+3}u + 3u\partial_x u - 2\partial_x u\partial_x^2 u - u\partial_x^3 u = 0. \tag{3.4}$$

Then, by applying $\cosh(\sigma|D|)$ to (3.4), we obtain the following equation for v_σ

$$\begin{aligned} \partial_t v_\sigma - \partial_t\partial_x^2 v_\sigma + (-1)^{j+1}\partial_x^{2j+1}v_\sigma + (-1)^{j+2}\partial_x^{2j+3}v_\sigma + 3v_\sigma\partial_x v_\sigma \\ -2\partial_x v_\sigma \partial_x^2 v_\sigma - v_\sigma\partial_x^3 v_\sigma = N(v_\sigma). \end{aligned} \tag{3.5}$$

where

$$N(v_\sigma) = \sum_{k=1}^{3} N_k(v_\sigma), \tag{3.6}$$

[3]Integration by parts is justified, since we may assume that $v_\sigma(x,t)$ and all of its spatial derivatives decays to zero as $|x| \to +\infty$ [18, see the foot note on page 9].

with

$$N_1(v_\sigma) = \frac{3}{2}\partial_x\left(v_\sigma^2 - \cosh(\sigma|D|)(\operatorname{sech}(\sigma|D|)v_\sigma)^2\right),$$
$$N_2(v_\sigma) = 2\left(\partial_x v_\sigma \partial_x^2 v_\sigma - \cosh(\sigma|D|\left[\partial_x(\operatorname{sech}(\sigma|D|)v_\sigma)\cdot\partial_x^2(\operatorname{sech}(\sigma|D|)v_\sigma)\right]\right),$$
$$N_3(v_\sigma) = \left(v_\sigma\partial_x^3 v_\sigma - \cosh(\sigma|D|)\left[\operatorname{sech}(\sigma|D|)v_\sigma\cdot\partial_x^3(\operatorname{sech}(\sigma|D|)v_\sigma)\right]\right).$$

By differentiating $E_\sigma(t)$ with respect to t and applying equations (3.5), (3.6), along with integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}E_\sigma(t) &= \int_{\mathbb{R}} (v_\sigma\partial_t v_\sigma + \partial_x v_\sigma \partial_t\partial_x v_\sigma)\,dx\\ &= \int_{\mathbb{R}} v_\sigma\cdot\left(1-\partial_x^2\right)\partial_t v_\sigma dx\\ &= \int_{\mathbb{R}} v_\sigma\cdot(M(v_\sigma)+N(v_\sigma))\,dx\\ &= \int_{\mathbb{R}} v_\sigma\cdot N(v_\sigma)dx,\end{aligned}$$

where $N(v_\sigma)$ is defined in equation (3.6) and $M(v_\sigma)$ is given by

$$M(v_\sigma) := (-1)^{j+2}\partial_x^{2j+1}v_\sigma + (-1)^{j+3}\partial_x^{2j+3}v_\sigma - 3v_\sigma\partial_x v_\sigma + 2\partial_x v_\sigma\partial_x^2 v_\sigma + v_\sigma\partial_x^3 v_\sigma.$$

Subsequently, integrating this result over the time interval $(0,\tau)$ for $\tau \leqslant \delta$ yields the desired equality (3.2). □

The integral (3.3) satisfies the following a priori estimate.

Lemma 3.1. *Let $\delta > 0$ and $1/2 < b \leqslant 1$ and $6 \leqslant j \leqslant 10$ be any integer. Then for any $v_\sigma \in X_\delta^{1,b}$ and $0 \leqslant \tau \leqslant \delta$, we have*

$$\sup_{\tau\in[0,\delta]} |R_\sigma(\tau)| \leqslant C\sigma^{\frac{3}{2}}\|v_\sigma\|^3_{X_\delta^{1,b}}, \tag{3.7}$$

for a constant $C > 0$ depending on δ.

To prove this Lemma, we will write $R_\sigma(\tau)$ as

$$R_\sigma(\tau) = \underbrace{\int_0^\tau\int_{\mathbb{R}} v_\sigma\cdot N_1(v_\sigma)dxdt}_{:=R_\sigma^{(1)}(\tau)} + \underbrace{\int_0^\tau\int_{\mathbb{R}} v_\sigma\cdot N_2(v_\sigma)dxdt}_{:=R_\sigma^{(2)}(\tau)} + \underbrace{\int_0^\tau\int_{\mathbb{R}} v_\sigma\cdot N_3(v_\sigma)dxdt}_{:=R_\sigma^{(3)}(\tau)}, \tag{3.8}$$

where $N_k(v_\sigma)$ for $k = 1,2,3$ are as in (3.6).

Thus, the integrals $R_\sigma^{(k)}(\tau)$ for $k = 1,2,3$ satisfy the following estimates.

$$\sup_{\tau\in[0,\delta]}\left|R_\sigma^{(1)}(\tau)\right| \lesssim \sigma^{2\gamma}\|v_\sigma\|^3_{X_\delta^{1,b}} \quad \text{for } \gamma\in\left[\frac{3}{4},1\right], \tag{3.9}$$

$$\sup_{\tau\in[0,\delta]}\left|R_\sigma^{(2)}(\tau)\right| \lesssim \sigma^{2\gamma}\|v_\sigma\|^3_{X_\delta^{1,b}} \quad \text{for } \gamma\in\left[\frac{3}{4},\frac{7}{8}\right], \tag{3.10}$$

$$\sup_{\tau\in[0,\delta]}\left|R_\sigma^{(3)}(\tau)\right| \lesssim \sigma^{\frac{3}{2}}\|v_\sigma\|^3_{X_\delta^{1,b}}, \tag{3.11}$$

when $j=6$ and

$$\sup_{\tau\in[0,\delta]}\left|R_\sigma^{(k)}(\tau)\right| \lesssim \sigma^{2}\|v_\sigma\|^3_{X_\delta^{1,b}}, \quad k=1,2,3, \tag{3.12}$$

when $7\leqslant j\leqslant 10$.

Proof of (3.9). By Plancherel's Theorem and Lemma 2.3, we have

$$\begin{aligned}\left|R_\sigma^{(1)}(\tau)\right| &= \left|\int_0^\tau\int_{\mathbb{R}} \widehat{v_\sigma}(\xi,t)\cdot\widehat{N_1(v_\sigma)}(\xi,t)\,d\xi dt\right| \\ &\leqslant \int_0^\delta\int_{\mathbb{R}^3} |\widehat{v_\sigma}(\xi,t)|\cdot|\xi||K(\sigma\xi)|\prod_{m=1}^2|\widehat{v_\sigma}(\xi_m,t)|d\mu(\xi)dt,\end{aligned}$$

where $K(\sigma|\xi|)$ is as in Lemma 2.3 and $d\mu(\xi)$ is the measure given by

$$d\mu(\xi) := \delta(\xi-\xi_1+\xi_2)d\xi_1 d\xi_2. \tag{3.13}$$

This imposes the condition $\xi=\xi_1-\xi_2$.

Clearly, we have

$$|K(\sigma\xi)|\leqslant 1. \tag{3.14}$$

By symmetry, we can assume that $|\xi_1|\leqslant|\xi_2|$. Then, we have

$$|K(\sigma\xi)|\leqslant C\sigma^2|\xi_1||\xi_2|. \tag{3.15}$$

For any $0\leqslant\gamma\leqslant 1$, interpolation between (3.14) and (3.15) gives

$$|K(\sigma\xi)|\leqslant C\sigma^{2\gamma}|\xi_1|^\gamma|\xi_2|^\gamma. \tag{3.16}$$

In particular, choosing $\gamma=\frac{3}{4}$ yields

$$|K(\sigma\xi)|\leqslant C\sigma^{\frac{3}{2}}|\xi_1|^{\frac{3}{4}}|\xi_2|^{\frac{3}{4}}. \tag{3.17}$$

By symmetry, we may assume that $|\xi_1|\leqslant|\xi_2|$, which implies $|\xi|\leqslant 2|\xi_2|$. Consequently, let's denote

$$\widehat{w_\sigma}=|\widehat{v_\sigma}|. \tag{3.18}$$

For any choice of $\gamma\in[\frac{3}{4},1]$ and $\alpha\in[\frac{3}{2},4]$, applying Plancherel's Theorem, (3.16), Hölder's inequality, (2.13), (2.16) and Sobolev embedding, we obtain

$$\begin{aligned}\left|R_\sigma^{(1)}(\tau)\right| &\lesssim \sigma^{2\gamma}\int_0^\delta\int_{\mathbb{R}^3}|\xi|\widehat{w_\sigma}(\xi,t)\cdot|\xi_1|^\gamma|\xi_2|^\gamma\prod_{m=1}^2\widehat{w_\sigma}(\xi_m,t)d\mu(\xi)dt \\ &\lesssim \sigma^{2\gamma}\int_0^\delta\int_{\mathbb{R}^3}\widehat{w_\sigma}(\xi,t)\cdot\langle\xi_1\rangle^\gamma\widehat{w_\sigma}(\xi_1,t)\cdot\langle\xi_2\rangle^{1+\gamma}\widehat{w_\sigma}(\xi_2,t)d\mu(\xi)dt \\ &\lesssim \sigma^{2\gamma}\int_0^\delta\int_{\mathbb{R}} w_\sigma\cdot\langle D\rangle w_\sigma\cdot\langle D\rangle^{1+\gamma}w_\sigma\,dxdt\end{aligned}$$

$$\lesssim \delta^{\frac{25-\alpha}{26}} \sigma^{2\gamma} \|w_\sigma\|_{L^\infty_\delta L^2_x} \|\langle D\rangle w_\sigma\|_{L^\infty_\delta L^2_x} \left\|\langle D\rangle^{1+\gamma} w_\sigma\right\|_{L_\delta^{\frac{26}{1+\alpha}} L^\infty_x}$$
$$\lesssim \sigma^{2\gamma} \|w_\sigma\|^3_{X^{1,b}_\delta} \sim \sigma^{2\gamma} \|v_\sigma\|^3_{X^{1,b}_\delta},$$

where the second line follows from the fact that $|\xi| \leqslant \langle\xi\rangle$ for any $\xi \in \mathbb{R}$. □

Proof of (3.10). By Plancherel's Theorem and Lemma 2.3, we have

$$\left|R^{(2)}_\sigma(\tau)\right| = \left|\int_0^\tau \int_{\mathbb{R}} \widehat{v_\sigma}(\xi, t) \cdot \widehat{N_2(v_\sigma)}(\xi, t) d\xi dt\right|$$
$$\leqslant \int_0^\delta \int_{\mathbb{R}^3} |\widehat{v_\sigma}(\xi, t)| \cdot |K(\sigma\xi)|\, |\xi_1||\xi_2|^2 \prod_{m=1}^{2} |\widehat{v_\sigma}(\xi_m, t)| d\mu(\xi) dt,$$

where $d\mu(\xi)$ and $K(\sigma\xi)$ be as in (3.13) and Lemma 2.3, respectively.

For any choice of $\gamma \in [\frac{3}{4}, \frac{7}{8}]$ and $\alpha \in [\frac{3}{2}, \frac{11}{2}]$, applying Plancherel's Theorem, (3.16), Hölder's inequality, (2.13), (2.16) and Sobolev embedding, we obtain

$$\left|R^{(2)}_\sigma(\tau)\right| \lesssim \sigma^{2\gamma} \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi, t) \cdot |\xi_1|^\gamma |\xi_2|^\gamma \cdot |\xi_1||\xi_2|^2 \prod_{m=1}^{2} \widehat{w_\sigma}(\xi_m, t) d\mu(\xi) dt$$
$$\lesssim \sigma^{2\gamma} \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi, t) \cdot \langle\xi_1\rangle \widehat{w_\sigma}(\xi_1, t) \cdot \langle\xi_2\rangle^{2+2\gamma} \widehat{w_\sigma}(\xi_2, t) d\mu(\xi) dt$$
$$\lesssim \sigma^{2\gamma} \int_0^\delta \int_{\mathbb{R}} w_\sigma \cdot \langle D\rangle w_\sigma \cdot \langle D\rangle^{2+2\gamma} w_\sigma dx dt$$
$$\lesssim \delta^{\frac{25-\alpha}{26}} \sigma^{2\gamma} \|w_\sigma\|_{L^\infty_\delta L^2_x} \|\langle D\rangle w_\sigma\|_{L^\infty_\delta L^2_x} \left\|\langle D\rangle^{2+2\gamma} w_\sigma\right\|_{L_\delta^{\frac{26}{1+\alpha}} L^\infty_x}$$
$$\lesssim \sigma^{2\gamma} \|w_\sigma\|^3_{X^{1,b}_\delta} \sim \sigma^{2\gamma} \|v_\sigma\|^3_{X^{1,b}_\delta}.$$

□

proof of (3.11). By Plancherel's Theorem and Lemma 2.3, we have

$$\left|R^{(3)}_\sigma(\tau)\right| = \left|\int_0^\tau \int_{\mathbb{R}} \widehat{v_\sigma}(\xi, t) \cdot \widehat{N_3(v_\sigma)}(\xi, t) d\xi dt\right|$$
$$\leqslant \int_0^\delta \int_{\mathbb{R}^3} |\widehat{v_\sigma}(\xi, t)| \cdot |K(\sigma\xi)|\, |\xi_3|^3 \prod_{m=1}^{2} |\widehat{v_\sigma}(\xi_m, t)| d\mu(\xi) dt,$$

where $d\mu(\xi)$ and $K(\sigma\xi)$ are as defined in (3.13) and Lemma 2.3, respectively.

Applying Plancherel's Theorem, (3.17), Hölder's inequality, (2.14), and Sobolev embedding, we obtain

$$\left|R^{(3)}_\sigma(\tau)\right| \lesssim \sigma^{\frac{3}{2}} \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi, t) \cdot |\xi_1|^{\frac{3}{4}} |\xi_2|^{\frac{3}{4}} \cdot |\xi_2|^3 \prod_{m=1}^{2} \widehat{w_\sigma}(\xi_m, t) d\mu(\xi) dt$$

$$\lesssim \sigma^{\frac{3}{2}} \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi, t) \cdot \langle \xi_1 \rangle^{\frac{3}{4}} \widehat{w_\sigma}(\xi_1, t) \cdot \langle \xi_2 \rangle^{\frac{15}{4}} \widehat{w_\sigma}(\xi_2, t) d\mu(\xi) dt$$
$$\lesssim \sigma^{\frac{3}{2}} \int_0^\delta \int_{\mathbb{R}} w_\sigma \cdot \langle D \rangle^{\frac{3}{4}} w_\sigma \cdot \langle D \rangle^{\frac{15}{4}} w_\sigma dx dt$$
$$\lesssim \delta^{\frac{3}{4}} \sigma^{\frac{3}{2}} \|w_\sigma\|_{L^\infty_\delta L^2_x} \left\| \langle D \rangle^{\frac{3}{4}} w_\sigma \right\|_{L^\infty_\delta L^2_x} \left\| \langle D \rangle^{\frac{15}{4}} w_\sigma \right\|_{L^4_\delta L^\infty_x}$$
$$\lesssim \sigma^{\frac{3}{2}} \|w_\sigma\|^3_{X^{1,b}_\delta} \sim \sigma^{\frac{3}{2}} \|v_\sigma\|^3_{X^{1,b}_\delta}.$$

□

Therefore, we have

$$|R_\sigma(\tau)| \leqslant |R^{(1)}_\sigma(\tau)| + |R^{(2)}_\sigma(\tau)| + |R^{(3)}_\sigma(\tau)|,$$

and hence we obtain $|R_\sigma(\tau)| \lesssim \sigma^{\frac{3}{2}} \|v_\sigma\|^3_{X^{1,b}_\delta}$.

Proof of (3.12). We prove the desired estimate of $R_\sigma(\tau)$ by estimating the quantities $R^{(n)}_\sigma(\tau)$ for $n = 1, 2, 3$ as follows:

Estimate for $R^{(1)}_\sigma(\tau)$. By Plancherel's Theorem and Lemma 2.3, we have

$$\left| R^{(1)}_\sigma(\tau) \right| = \left| \int_0^\tau \int_{\mathbb{R}} \widehat{v_\sigma}(\xi, t) \cdot \widehat{N_1(v_\sigma)}(\xi, t) d\xi dt \right|$$
$$\leqslant \int_0^\delta \int_{\mathbb{R}^3} |\widehat{v_\sigma}(\xi, t)| \cdot |\xi| |K(\sigma\xi)| \prod_{m=1}^{2} |\widehat{v_\sigma}(\xi_m, t)| d\mu(\xi) dt,$$

where $d\mu(\xi)$ is as given in (3.13).

Denoting

$$\widehat{w_\sigma} = |\widehat{v_\sigma}|. \tag{3.19}$$

The, we obtain from the Plancherel's Theorem, Hölder's inequality, (3.15), (2.14) for $\alpha = 2$, (2.8) and one dimensional Sobolev embedding that

$$\left| R^{(1)}_\sigma(\tau) \right| \lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}^3} |\xi| \widehat{w_\sigma}(\xi, t) \cdot |\xi_1| |\xi_2| \prod_{m=1}^{2} \widehat{w_\sigma}(\xi_m, t) d\mu(\xi) dt$$
$$\lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi, t) \cdot \langle \xi_1 \rangle \widehat{w_\sigma}(\xi_1, t) \cdot \langle \xi_2 \rangle^2 \widehat{w_\sigma}(\xi_2, t) d\mu(\xi) dt$$
$$\lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}} w_\sigma \cdot \langle D \rangle w_\sigma \cdot \langle D \rangle^2 w_\sigma \; dx dt$$
$$\lesssim \delta^{\frac{3}{4}} \sigma^2 \|w_\sigma\|_{L^\infty_\delta L^2_x} \|\langle D \rangle w_\sigma\|_{L^\infty_\delta L^2_x} \left\| \langle D \rangle^2 w_\sigma \right\|_{L^4_\delta L^\infty_x}$$

$$\lesssim \sigma^2 \|w_\sigma\|^3_{X^{1,b}_\delta} \sim \sigma^2 \|v_\sigma\|^3_{X^{1,b}_\delta}.$$

□

Estimate for $R^{(2)}_\sigma(\tau)$. By Plancherel's Theorem and Lemma 2.3, we have

$$\begin{aligned}\left|R^{(2)}_\sigma(\tau)\right| &= \left|\int_0^\tau \int_{\mathbb{R}} \widehat{v_\sigma}(\xi,t) \cdot \widehat{N_2(v_\sigma)}(\xi,t)\,d\xi dt\right| \\ &\leqslant \int_0^\delta \int_{\mathbb{R}^3} |\widehat{v_\sigma}(\xi,t)| \cdot |K(\sigma\xi)|\,|\xi_1||\xi_2|^2 \prod_{m=1}^{2} |\widehat{v_\sigma}(\xi_m,t)| d\mu(\xi) dt,\end{aligned}$$

where $d\mu(\xi)$ and $K(\sigma\xi)$ be as in (3.13) and Lemma 2.3, respectively.

By using symmetry, Plancherel's Theorem, (3.15), Hölder's inequality, (2.14) for $\alpha = 6$, (2.8) and Sobolev embedding, we obtain

$$\begin{aligned}\left|R^{(2)}_\sigma(\tau)\right| &\lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi,t) \cdot |\xi_1||\xi_2| \cdot |\xi_1||\xi_2|^2 \prod_{m=1}^{2} \widehat{w_\sigma}(\xi_m,t) d\mu(\xi) dt \\ &\lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi,t) \cdot \langle\xi_1\rangle \widehat{w_\sigma}(\xi_1,t) \cdot \langle\xi_2\rangle^4 \widehat{w_\sigma}(\xi_2,t) d\mu(\xi) dt \\ &\lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}} w_\sigma \cdot \langle D\rangle w_\sigma \cdot \langle D\rangle^4 w_\sigma \,dx dt \\ &\lesssim \delta^{\frac{3}{4}} \sigma^2 \|w_\sigma\|_{L^\infty_\delta L^2_x} \|\langle D\rangle w_\sigma\|_{L^\infty_\delta L^2_x} \left\|\langle D\rangle^4 w_\sigma\right\|_{L^4_\delta L^\infty_x} \\ &\lesssim \sigma^2 \|w_\sigma\|^3_{X^{1,b}_\delta} \sim \sigma^2 \|v_\sigma\|^3_{X^{1,b}_\delta}.\end{aligned}$$

□

Estimate for $R^{(3)}_\sigma(\tau)$. By Plancherel's Theorem and Lemma 2.3, we have

$$\begin{aligned}\left|R^{(3)}_\sigma(\tau)\right| &= \left|\int_0^\tau \int_{\mathbb{R}} \widehat{v_\sigma}(\xi,t) \cdot \widehat{N_3(v_\sigma)}(\xi,t)\,d\xi dt\right| \\ &\leqslant \int_0^\delta \int_{\mathbb{R}^3} |\widehat{v_\sigma}(\xi,t)| \cdot |K(\sigma\xi)|\,|\xi_3|^3 \prod_{m=1}^{2} |\widehat{v_\sigma}(\xi_m,t)| d\mu(\xi) dt,\end{aligned}$$

where $d\mu(\xi)$ and $K(\sigma\xi)$ be as in (3.13) and Lemma 2.3, respectively.

Applying Plancherel's Theorem, (3.15), Hölder's inequality, (2.14) for $\alpha = 6$, (2.8) and Sobolev embedding gives

$$\left|R^{(3)}_\sigma(\tau)\right| \lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi,t) \cdot |\xi_1||\xi_2| \cdot |\xi_2|^3 \prod_{m=1}^{2} \widehat{w_\sigma}(\xi_m,t) d\mu(\xi) dt$$

$$
\begin{aligned}
&\lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}^3} \widehat{w_\sigma}(\xi,t) \cdot \langle \xi_1 \rangle \widehat{w_\sigma}(\xi_1,t) \cdot \langle \xi_2 \rangle^4 \widehat{w_\sigma}(\xi_2,t) d\mu(\xi) dt \\
&\lesssim \sigma^2 \int_0^\delta \int_{\mathbb{R}} w_\sigma \cdot \langle D \rangle w_\sigma \cdot \langle D \rangle^4 w_\sigma dx dt \\
&\lesssim \delta^{\frac{3}{4}} \sigma^2 \|w_\sigma\|_{L^\infty_\delta L^2_x} \|\langle D \rangle w_\sigma\|_{L^\infty_\delta L^2_x} \left\| \langle D \rangle^4 w_\sigma \right\|_{L^4_\delta L^\infty_x} \\
&\lesssim \sigma^2 \|w_\sigma\|^3_{X^{1,b}_\delta} \sim \sigma^2 \|v_\sigma\|^3_{X^{1,b}_\delta}.
\end{aligned}
$$

□

Therefore, we have

$$
|R_\sigma(\tau)| \leqslant |R^{(1)}_\sigma(\tau)| + |R^{(2)}_\sigma(\tau)| + |R^{(3)}_\sigma(\tau)|,
$$

and hence we obtain $|R_\sigma(\tau)| \lesssim \sigma^{\frac{1}{2}} \|v_\sigma\|^3_{X^{1,b}_\delta}$.

□

Theorem 4 (Approximate conservation law)**.** *Let $u_0 \in H^{\sigma,1}$ for $\sigma > 0$. Suppose that u is the local solution to the IVP (1.1)–(1.2) on $\mathbb{R} \times [0,\delta]$ that is constructed in Theorem 3. Then,*

$$
\sup_{t \in [0,\delta]} E_\sigma(t) \leqslant E_{\sigma_0}(0) + C\sigma^{2\gamma} E^{\frac{3}{2}}_\sigma(0), \tag{3.20}
$$

for some constant $C > 0$, where $\gamma = \frac{3}{4}$ if $j = 6$ and $\gamma = 1$ if $7 \leqslant j \leqslant 10$ is any integer.

Proof. First, observe that by (2.18),

$$
\|v_\sigma\|_{X^{1,b}_\delta} = \|u\|_{X^{\sigma,1,b}_\delta} \leqslant C\|u_0\|_{H^{\sigma,1}} = \|v_\sigma(0)\|_{H^1}. \tag{3.21}
$$

On the other hand, we have

$$
\begin{aligned}
\|v_\sigma(\cdot,0)\|^2_{H^1} &= \int_{\mathbb{R}} \langle \xi \rangle^2 |\widehat{v_\sigma}(\xi,0)|^2 d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|^2) |\widehat{v_\sigma}(\xi,0)|^2 d\xi \\
&= \int_{\mathbb{R}} \left(|\widehat{v_\sigma}(\xi,0)|^2 + |\xi|^2 |\widehat{v_\sigma}(\xi,0)|^2 \right) d\xi \\
&= \int_{\mathbb{R}} |v_\sigma(x,0)|^2 dx + \int_{\mathbb{R}} |\partial_x v_\sigma(x,0)|^2 dx \\
&= 2E_{\sigma_0}(0),
\end{aligned}
$$

which in turn implies

$$
\|v_\sigma(\cdot,0)\|_{H^1} = 2^{\frac{1}{2}} \sqrt{E_{\sigma_0}(0)}. \tag{3.22}
$$

Then, (3.22) combined with (3.21) gives

$$
\|v_\sigma\|_{X^{1,b}_\delta} \leqslant C\sqrt{E_{\sigma_0}(0)}. \tag{3.23}
$$

Finally, the estimate (3.23) together with (3.7) yields (3.20). □

4. Proof of Theorem 2

Assume that $u(\cdot,0)=u_0(\cdot)\in H^{\sigma_0,1}$ for some $\sigma_0>0$, which in turn implies

$$v_{\sigma_0}(\cdot,0)=\cosh(\sigma_0|D|)u_0(\cdot)\in H^1.$$

Then from the definitions of a modified energy, we have

$$E_{\sigma_0}(0)=\frac{1}{2}\|v_{\sigma_0}(\cdot,0)\|^2_{H^1}<\infty.$$

Following the argument in [12, 18] (see also [7]), we can construct a solution on $[0,T]$ for arbitrarily large time T. We achieve this by applying the approximate conservation law in Theorem 4 and repeating the local-in-time result on consecutive short intervals $[0,\delta]$. This process allows us to adjust the strip width parameter $\sigma\in[0,\sigma_0]$ of the solution according to the size of T. To begin, we first note that by Theorem 4,

$$\begin{aligned}\sup_{0\leqslant t\leqslant\delta_0} E_\sigma(t) &\leqslant E_\sigma(0)+c\sigma^{2\gamma}E_\sigma^{\frac{3}{2}}(0)\\ &\leqslant E_{\sigma_0}(0)+c\sigma^{2\gamma}E_{\sigma_0}^{\frac{3}{2}}(0),\end{aligned}\tag{4.1}$$

for some δ_0 in the interval $(0,\delta]$, where γ be as in Theorem 4. Here, to get the second line we used the fact that $E_\sigma(0)\leqslant E_{\sigma_0}(0)$ which holds for $\sigma\leqslant\sigma_0$ as $\cosh r$ is increasing for $r\geqslant 0$. Thus,

$$\sup_{0\leqslant t\leqslant\delta_0} E_\sigma(t)\leqslant 2E_{\sigma_0}(0)\tag{4.2}$$

provided that

$$c\sigma^{2\gamma}E_{\sigma_0}^{\frac{3}{2}}(0)\leqslant E_{\sigma_0}(0).\tag{4.3}$$

Next, we apply the local theory with initial time $t=\delta_0$ and time-step size δ to extend the solution from $[0,\tau]$ to $[\tau,\tau+\delta]$. By Theorem 3.20 and (4.2) we obtain

$$\sup_{\delta_0\leqslant t\leqslant\delta_0+\delta} E_\sigma(t)\leqslant E_\sigma(\delta_0)+c\sigma^{2\gamma}2^{\frac{3}{2}}E_\sigma^{\frac{3}{2}}(0).\tag{4.4}$$

Proceeding in this manner we can cover all time intervals $[0,\delta]$, $[\delta,2\delta]$, $[2\delta,3\delta]$, etc., and then apply induction (see e.g, [12]) to establish

$$\sup_{0\leqslant t\leqslant T} E_\sigma(t)\leqslant 2E_{\sigma_0}(0)\quad\text{for}\quad \sigma\geqslant cT^{-\frac{1}{2\gamma}},\tag{4.5}$$

where $c>0$ depends on $E_{\sigma_0}(0)$. This would in turn imply

$$\sup_{0\leqslant t\leqslant T}\|u(t)\|_{H^{\sigma,1}(\mathbb{R})}<\infty\quad\text{for}\quad\sigma\geqslant cT^{-\frac{1}{2\gamma}}$$

which proves Theorem 2.

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The authors have no conflicts of interest to disclose.

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