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SOLUTIONS OF CAPUTO K-FRACTIONAL DIFFERENTIAL EQUATIONS USING FRACTIONAL FOURIER TRANSFORM

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ABSTRACT. The aim of this paper is to study the extended generalized Mittag-Leffler function. Further, the Fractional Fourier transform of the Caputo k-fractional derivative is presented. Caputo k-fractional differential equations are presented and their solutions are proposed by applying the fractional Fourier transform.

Keywords: Fractional differential equation, Fractional derivative, Caputo k-fractional derivative, Mittag-Leffler function, Fractional Fourier transform.

1. Introduction

Fractional differential equations have recently become a very strong tool in many fields such as Physics, Thermodynamics, Electrical circuit theory and fractalness, Seepage flow in porous media, Mechatronics systems, signal processing, Chemical mixing, Chaos theory, etc. Fractional differential equations and its applications are discussed up to date in books of Oldharam and Spannier Sanko kilbas, Marcher, Miller and Ross, Podlubry. The most important advantage of using a fractional differential equation is their non-local property that is the next state of a system depend not only upon its current state but also upon all of its historical states. when describing differential, fractional and functional equations as physical models and practical problems, finding exact solutions to these equations is improbably difficult. If there exist exact solutions is often so complicated that is not convenient for numerical solutions. In view of this, it is necessary to discuss approximate solutions that lie near-exact solutions.
Swedish mathematician Magnus Gosta Mittag-Leffler defined a new special function for divergent series which is known as Mittag-Leffler (ML) function in 1948. Pollard derived one parameter Mittag-Leffler function. Main features of ML function and generalization of two-parameter Mittag-Leffler function derived by P. Humbert, et al.. K.S. Cole, B. Gross, M. Caputo, and F. Mandari gave their crucial contributions to fractional calculus and showing the Mittag-Leffler function always appears in the material functions of a system described by fractional-order equations.ML function was named the Queen function of the fractional calculus by F. Mandari and R. Gorenflo.
Special praise is due to the three-parameter extension of the ML function introduced by the Indian mathematician Tilak Raj Prabhakar in [31] dating back to

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1971.
Fourier transform (FT) was defined by French mathematician Joseph Fourier. Fractional Fourier transforms (FrFT) is the generalization of the conventional Fourier transform. In 1980 Victor Nami initially introduced the FrFT gives the best outcomes for non-stationary signals than Fourier Transform due to its time-frequency characteristics. FrFT is used in several scientific fields such as optics, time-frequency distribution, image processing, satellite image compression signal image recovery and noise removal, image smoothing, encryption, and decryption. V. Namias as a way to solve FrFT plays a very important role in solving ordinary and partial equations
Generally Transforms play very important role in many scientific field because it transmits signal without loss of information.
An Integral transform maps a function its original function space into another function space via integration, where some properties and characteristic of original function manipulated.
The transformed function back to original function space using inverse transform. FractionalFourier transform was developed to remove the requirement of finite intervals.

2. Preliminaries

Definition 2.1. (The Riemann–Liouville Fractional Differential Operator)
Suppose that $n-1 \leq \alpha < n$. Then,

$$^{RL}D^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau \tag{2.1}$$

is called the Riemann–Liouville derivative operator of order α.

Definition 2.2. (The Caputo Fractional Differential Operator)
Suppose that $n-1 < \alpha \leq n$. Then,

$$^{C}D^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau \tag{2.2}$$

is called the Caputo derivative operator of order α.

Definition 2.3. (The Mittag–Leffler Function)
The one and two-parameter representations of the Mittag–Leffler function can be defined in terms of a power series as:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad (\alpha > 0) \tag{2.3}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha, \beta > 0) \tag{2.4}$$

The exponential series defined by (**??**) gives the generalization of (**??**); it was introduced by R.P. Agarwal in 1953.

Definition 2.4. (Prabhakar Function)
In 1970, the generalized Mittag–Leffler function with three parameters was introduced by Prabhakar as:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\alpha k + \beta)} \tag{2.5}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\text{Re}(\alpha) > 0$. Here, $(\gamma)_k$ is the Pochhammer symbol defined as:

$$(\gamma)_k = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} \tag{2.6}$$

This function is also known as the Prabhakar function.

Definition 2.5. (Prabhakar Integral)
For $\text{Re}(\alpha) > 0$, the Prabhakar integral operator including the generalized Mittag–Leffler function is defined as follows:

$$\left(\mathbf{E}^{\gamma}_{\alpha,\beta,\omega,a+} f\right)(t) = \int_a^t (t-\tau)^{\beta-1} E^{\gamma}_{\alpha,\beta}\left[\omega(t-\tau)^{\alpha}\right] f(\tau) d\tau \tag{2.7}$$

where $\omega \in \mathbb{R}$.

For $\gamma = 0$, the Prabhakar integral operator coincides with the Riemann–Liouville fractional integral of order β (or μ):

$$\mathbf{E}^{0}_{\alpha,\beta,\omega,a+} f(t) = I^{\beta}_{a+} f(t) \tag{2.8}$$

Definition 2.6. (Prabhakar Derivative)
For $m - 1 < \mu \leq m$, the Prabhakar derivative is defined by:

$${}^{P}D^{\gamma}_{\alpha,\mu,\omega,a+} f(t) = D^m \left[\mathbf{E}^{-\gamma}_{\alpha,m-\mu,-\omega,a+} f(t)\right] \tag{2.9}$$

where $D^m = \frac{d^m}{dt^m}$.

It is obvious that the Prabhakar fractional derivative generates the Riemann–Liouville fractional derivative when parameters are specialized:

$${}^{P}D^{0}_{\alpha,\mu,0,a+} f(t) = {}^{RL}D^{\mu}_{a+} f(t) \tag{2.10}$$

Lemma 2.7. *Let $Re(\alpha) > 0$. Then for any $n \in \mathbb{N}$, the differentiation of the generalized Mittag–Leffler function is given by:*

$$\left(\frac{d}{dz}\right)^n E^{\gamma}_{\alpha,\beta}(z) = (\gamma)_n E^{\gamma+n}_{\alpha,\beta+n\alpha}(z) \tag{2.11}$$

Lemma 2.8. *The following series including the generalized Mittag–Leffler function with three parameters is an absolutely convergent series:*

$$\sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} z^k = (1-z)^{-\gamma}, \quad |z| < 1 \tag{2.12}$$

Definition 2.9. (k-Riemann–Liouville Fractional Integrals)
Let $k > 0$ and $\mu > 0$. Then the left-sided and right-sided k-Riemann–Liouville fractional integrals of order μ for a function g are defined by:

$${}^{k}I^{\mu}_{a+} g(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} g(t) dt \tag{2.13}$$

and

$$ {}^{k}I^{\mu}_{b-}g(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} g(t)dt \tag{2.14} $$

where Γ_k is the k-gamma function defined as:

$$ \Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt \tag{2.15} $$

Also, it satisfies the identity $\Gamma_k(z+k) = z\Gamma_k(z)$.

Definition 2.10. (k-Caputo Fractional Derivatives)
Let $k > 0$ and $n-1 < \mu \leq n$. Then the left and right-sided Caputo k-fractional derivatives of order μ are defined by:

$$ {}^{C,k}D^{\mu}_{a+}g(x) = \frac{1}{k\Gamma_k(nk-\mu)} \int_a^x (x-t)^{\frac{nk-\mu}{k}-1} g^{(n)}(t)dt \tag{2.16} $$

and

$$ {}^{C,k}D^{\mu}_{b-}g(x) = \frac{(-1)^n}{k\Gamma_k(nk-\mu)} \int_x^b (t-x)^{\frac{nk-\mu}{k}-1} g^{(n)}(t)dt \tag{2.17} $$

Lemma 2.11. *Let $k > 0$ and $n-1 < \mu \leq n$. Then the left-sided Caputo k-fractional derivative is given by:*

$$ {}^{C,k}D^{\mu}_{a+}g(x) = {}^{k}I^{nk-\mu}_{a+}g^{(n)}(x) \tag{2.18} $$

Lemma 2.12. *(Lemma 2.10)*
Suppose that $k > 0$ and $n-1 < \mu \leq n$. If there exists the limit

$$ \lim_{x\to a^+} g^{(i)}(x) = 0, \quad i = 0, 1, 2, \ldots, n-1 \tag{2.19} $$

Then the following relation holds:

$$ {}^{k}I^{\mu}_{a+}\left({}^{C,k}D^{\mu}_{a+}g(x)\right) = g(x) \tag{2.20} $$

Also, if $g \in C^n[a,b]$ and there exists the limit $\lim_{x\to a^+} g^{(i)}(a) = c_i$, then:

$$ {}^{k}I^{\mu}_{a+}\left({}^{C,k}D^{\mu}_{a+}g(x)\right) = g(x) - \sum_{i=0}^{n-1} \frac{g^{(i)}(a)}{i!}(x-a)^i \tag{2.21} $$

Definition 2.13. (Fractional Fourier Transform)
For a function u, the fractional Fourier transform of order α (where $\phi = \alpha\pi/2$) is defined as:

$$ \mathcal{F}_\alpha[u](\xi) = \int_{-\infty}^{\infty} u(t)K_\alpha(t,\xi)dt \tag{2.22} $$

where the kernel $K_\alpha(t,\xi)$ is given by:

$$ K_\alpha(t,\xi) = \sqrt{\frac{1-i\cot\phi}{2\pi}} \exp\left(i\frac{t^2+\xi^2}{2}\cot\phi - it\xi\csc\phi\right) \tag{2.23} $$

If $\alpha = 1$, the kernel $K_1(t,\xi)$ coincides with the conventional Fourier transform kernel $\frac{1}{\sqrt{2\pi}}e^{-it\xi}$.

Definition 2.14. (One-to-one)
Let f, g be continuous and $\mathcal{F}_\alpha$ be piece-wise continuous. If $\mathcal{F}_\alpha[f] = \mathcal{F}_\alpha[g]$, then $f(x) = g(x)$ for every x.

Definition 2.15. (Convolution)
The fractional Fourier transform of the convolution $f *_\alpha g$ is related to the product of their individual transforms. Let $h = f *_\alpha g$, then:

$$\mathcal{F}_\alpha[f *_\alpha g](\xi) = e^{-i\frac{\xi^2}{2}\cot\phi}\mathcal{F}_\alpha[f](\xi) \cdot \mathcal{F}_\alpha[g](\xi) \tag{2.24}$$

The convolution in the time domain is equivalent to multiplication in the frequency domain (subject to a phase factor).

Let $f \in L^2(\mathbb{R})$, then the transform satisfies the Parseval relation:

$$\|\mathcal{F}_\alpha[f]\|_{L^2} = \|f\|_{L^2} \tag{2.25}$$

Theorem 2.16. *(Theorem 2.15)*
Let $f, g \in L^2(\mathbb{R})$, then the following holds:

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \int_{-\infty}^{\infty} \mathcal{F}_\alpha[f](\xi)\overline{\mathcal{F}_\alpha[g](\xi)}d\xi \tag{2.26}$$

3. Mathematical Preliminaries

Lemma 3.1. *The Fractional Fourier Transform (FrFT) of the Prabhakar integral operator $\mathbf{E}^\gamma_{\rho,\mu,\omega,a+}$ for a function $f(t)$ is defined as:*

$$\mathcal{F}_\alpha\left\{\left(\mathbf{E}^\gamma_{\rho,\mu,\omega,a+}f\right)(t)\right\}(u) = \mathcal{K}_\alpha(u,t)\cdot\hat{f}(u)\left[1-\omega(iu\sin\alpha+\cos\alpha)^{-\rho}\right]^{-\gamma} \tag{3.1}$$

where $\mathcal{K}_\alpha(u,t)$ is the FrFT kernel, and $\rho, \mu, \gamma \in \mathbb{C}$ with $Re(\rho) > 0, Re(\mu) > 0$.

4. Main Results

Lemma 4.1. *Let $k > 0$ and $n-1 < \beta \leq n$. Then the fractional Fourier transform of the left-sided Caputo k-fractional derivative ${}^C D^\beta_{k,a+}$ is given by:*

$$\mathcal{F}_\alpha\left\{{}^C D^\beta_{k,a+}f(t)\right\}(u) = \left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\beta/k}\mathcal{F}_\alpha\{f(t)\}(u) - \sum_{j=0}^{n-1}\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\frac{\beta-j-1}{k}} f^{(j)}(a+) \tag{4.1}$$

5. Mathematical Preliminaries

Lemma 5.1. *The Fractional Fourier Transform (FrFT) of the Prabhakar integral operator $\mathbf{E}^\gamma_{\rho,\mu,\omega,a+}$ for a function $f(t)$ is given by:*

$$\mathcal{F}_\alpha\left\{\left(\mathbf{E}^\gamma_{\rho,\mu,\omega,a+}f\right)(t)\right\}(u) = \mathcal{K}_\alpha(u,t)\cdot\hat{f}(u)\left[1-\omega(iu\sin\alpha+\cos\alpha)^{-\rho}\right]^{-\gamma} \tag{5.1}$$

where $\mathcal{K}_\alpha(u,t)$ denotes the FrFT kernel, and $\rho, \mu, \gamma \in \mathbb{C}$ such that $Re(\rho) > 0$ and $Re(\mu) > 0$.

6. Main Results

Lemma 6.1. *Let $k > 0$ and $n - 1 < \beta \leq n$. The Fractional Fourier Transform of the left-sided Caputo k-fractional derivative ${}^C D^{\beta}_{k,a+}$ is defined as:*

$$\mathcal{F}_\alpha \left\{ {}^C D^{\beta}_{k,a+} f(t) \right\}(u) = \left(\frac{iu \sin\alpha + \cos\alpha}{k} \right)^{\beta/k} \mathcal{F}_\alpha\{f(t)\}(u) - \sum_{j=0}^{n-1} \left(\frac{iu \sin\alpha + \cos\alpha}{k} \right)^{\frac{\beta - j - 1}{k}} f^{(j)}(a+) \tag{6.1}$$

7. Mathematical Preliminaries

Lemma 7.1. *The Fractional Fourier Transform (FrFT) of the Prabhakar integral operator $\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a+}$ for a function $f(t)$ is given by:*

$$\mathcal{F}_\alpha \left\{ \left(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a+} f \right)(t) \right\}(u) = \mathcal{K}_\alpha(u,t) \cdot \hat{f}(u) \left[1 - \omega (iu \sin\alpha + \cos\alpha)^{-\rho} \right]^{-\gamma} \tag{7.1}$$

where $\mathcal{K}_\alpha(u,t)$ denotes the FrFT kernel, and $\rho, \mu, \gamma \in \mathbb{C}$ such that $Re(\rho) > 0$ and $Re(\mu) > 0$.

8. Main Results

Lemma 8.1. *Let $k > 0$ and $n - 1 < \beta \leq n$. The Fractional Fourier Transform of the left-sided Caputo k-fractional derivative ${}^C D^{\beta}_{k,a+}$ is defined as:*

$$\mathcal{F}_\alpha \left\{ {}^C D^{\beta}_{k,a+} f(t) \right\}(u) = \left(\frac{iu \sin\alpha + \cos\alpha}{k} \right)^{\beta/k} \mathcal{F}_\alpha\{f(t)\}(u) - \sum_{j=0}^{n-1} \left(\frac{iu \sin\alpha + \cos\alpha}{k} \right)^{\frac{\beta - j - 1}{k}} f^{(j)}(a+) \tag{8.1}$$

Proof:
Consider the fractional differential equation:

$${}^C D^{\beta}_{k,a+} y(t) = f(t) \tag{1}$$

Applying the **Fractional Fourier Transform** (FrFT) with angle α to both sides of Eq. (1), we obtain:

$$\mathcal{F}_\alpha \left\{ {}^C D^{\beta}_{k,a+} y(t) \right\}(u) = \mathcal{F}_\alpha\{f(t)\}(u) \tag{8.2}$$

Using the operational property for the left-sided Caputo k-fractional derivative:

$$\left(\frac{iu \sin\alpha + \cos\alpha}{k} \right)^{\beta/k} \mathcal{F}_\alpha\{y(t)\}(u) - \sum_{j=0}^{n-1} \left(\frac{iu \sin\alpha + \cos\alpha}{k} \right)^{\frac{\beta - j - 1}{k}} y^{(j)}(a+) = \hat{f}_\alpha(u) \tag{8.3}$$

Rearranging the terms to isolate the transform of the unknown function $\mathcal{F}_\alpha\{y(t)\}(u)$

$$\left(\frac{iu \sin\alpha + \cos\alpha}{k} \right)^{\beta/k} \mathcal{F}_\alpha\{y(t)\}(u) = \hat{f}_\alpha(u) + \sum_{j=0}^{n-1} \left(\frac{iu \sin\alpha + \cos\alpha}{k} \right)^{\frac{\beta - j - 1}{k}} y^{(j)}(a+) \tag{8.4}$$

Dividing by the leading coefficient, we get:

$$\mathcal{F}_\alpha\{y(t)\}(u) = \frac{\hat{f}_\alpha(u)}{\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\beta/k}} + \sum_{j=0}^{n-1} \frac{\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\frac{\beta-j-1}{k}}}{\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\beta/k}} y^{(j)}(a+) \tag{8.5}$$

Simplifying the exponents in the summation term:

$$\mathcal{F}_\alpha\{y(t)\}(u) = \left(\frac{k}{iu\sin\alpha+\cos\alpha}\right)^{\beta/k} \hat{f}_\alpha(u) + \sum_{j=0}^{n-1} \left(\frac{k}{iu\sin\alpha+\cos\alpha}\right)^{\frac{j+1}{k}} y^{(j)}(a+) \tag{8.6}$$

This completes the transformation into the FrFT domain. Taking the inverse Fractional Fourier Transform on both sides of the transform domain equation, we obtain the general solution for $y(t)$:

$$y(t) = \mathcal{F}_{-\alpha}\left[\frac{\hat{f}_\alpha(u) + \sum_{j=0}^{n-1}\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\frac{\beta-j-1}{k}} y^{(j)}(a+)}{\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\beta/k}}\right] \tag{8.7}$$

9. Main Theorem

Theorem 9.1. *Let $k > 0$ with $n-1 < \beta \leq n$ and $f \in L^1[a,b]$. Suppose $y(t)$ is a function such that ${}^C D^\beta_{k,a+} y(t) \in C[a,b]$. Then, for $t \in [a,b]$, the fractional differential equation:*

$${}^C D^\beta_{k,a+} y(t) = f(t) \tag{9.1}$$

subject to the initial conditions:

$$y^{(j)}(a) = c_j, \quad j = 0, 1, \ldots, n-1 \tag{9.2}$$

where c_j are arbitrary constants, has the unique solution:

$$y(t) = \sum_{j=0}^{n-1} \frac{c_j}{\Gamma_k(jk+k)}(t-a)^j + \frac{1}{k\Gamma_k(\beta)} \int_a^t (t-\tau)^{\frac{\beta}{k}-1} f(\tau)d\tau \tag{9.3}$$

where Γ_k is the k-Gamma function defined as $\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-t^k/k}dt$.

Proof:
Consider the fractional differential equation:

$${}^C D^\beta_{k,a+} y(t) = f(t) \tag{1}$$

Taking the **Fractional Fourier Transform** (FrFT) with angle α on both sides of Eq. (1), we obtain:

$$\mathcal{F}_\alpha\left\{{}^C D^\beta_{k,a+} y(t)\right\}(u) = \mathcal{F}_\alpha\{f(t)\}(u) \tag{9.4}$$

By applying the operational property of the left-sided Caputo k-fractional derivative in the FrFT domain, we have:

$$\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\beta/k} \hat{y}_\alpha(u) - \sum_{j=0}^{n-1}\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\frac{\beta-j-1}{k}} y^{(j)}(a+) = \hat{f}_\alpha(u) \tag{9.5}$$

Isolating the transform of the unknown function $\hat{y}_\alpha(u)$:

$$\hat{y}_\alpha(u) = \frac{\hat{f}_\alpha(u)}{\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\beta/k}} + \sum_{j=0}^{n-1} \frac{\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\frac{\beta-j-1}{k}}}{\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\beta/k}} y^{(j)}(a+) \tag{9.6}$$

Simplifying the algebraic terms:

$$\hat{y}_\alpha(u) = \left(\frac{k}{iu\sin\alpha+\cos\alpha}\right)^{\beta/k} \hat{f}_\alpha(u) + \sum_{j=0}^{n-1} \left(\frac{k}{iu\sin\alpha+\cos\alpha}\right)^{\frac{j+1}{k}} y^{(j)}(a+) \tag{9.7}$$

Applying the **Inverse Fractional Fourier Transform** $\mathcal{F}_{-\alpha}$, we retrieve the solution in the time domain.

Theorem 9.2. *Let $k > 0$ with $n-1 < \beta \le n$ and $f \in L^1[a,b]$. Suppose $y(t)$ is such that ${}^C D^\beta_{k,a+} y(t) \in C[a,b]$. Then, for $t \in [a,b]$, the differential equation*

$${}^C D^\beta_{k,a+} y(t) = f(t) \tag{9.8}$$

with the initial conditions $y^{(j)}(a) = c_j$ for $j = 0, 1, \ldots, n-1$, where c_j are arbitrary constants, has the unique solution:

$$y(t) = \sum_{j=0}^{n-1} \frac{c_j}{\Gamma_k(jk+k)}(t-a)^j + \frac{1}{k\Gamma_k(\beta)} \int_a^t (t-\tau)^{\frac{\beta}{k}-1} f(\tau)d\tau \tag{9.9}$$

where $\Gamma_k(\cdot)$ denotes the k-Gamma function.

Taking the **Inverse Fractional Fourier Transform** (IFrFT) with angle $-\alpha$ on both sides of the transformed equation, we obtain the expression for $y(t)$ in the time domain:

$$y(t) = \mathcal{F}_{-\alpha}\left[\frac{\hat{f}_\alpha(u)}{\left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{\beta/k}} + \sum_{j=0}^{n-1} \left(\frac{iu\sin\alpha+\cos\alpha}{k}\right)^{-\frac{j+1}{k}} y^{(j)}(a+)\right] \tag{9.10}$$

10. Main Results

Theorem 10.1. *Let $k > 0$ with $n-1 < \beta \le n$ and $f \in L^1[a,b]$. Suppose $y(t)$ is a function such that ${}^C D^\beta_{k,a+} y(t) \in C[a,b]$. Then, for $t \in [a,b]$, the fractional differential equation*

$${}^C D^\beta_{k,a+} y(t) = f(t) \tag{10.1}$$

subject to the initial conditions:

$$y^{(j)}(a) = c_j, \quad j = 0, 1, \ldots, n-1 \tag{10.2}$$

where c_j are arbitrary constants, has the unique solution:

$$y(t) = \sum_{j=0}^{n-1} \frac{c_j}{\Gamma_k(jk+k)}(t-a)^j + \frac{1}{k\Gamma_k(\beta)} \int_a^t (t-\tau)^{\frac{\beta}{k}-1} f(\tau)d\tau \tag{10.3}$$

where $\Gamma_k(\cdot)$ denotes the k-Gamma function defined by $\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-t^k/k}dt$.

Proof

Step 1: Applying the Fractional Fourier Transform.
Applying the fractional Fourier Transform on both sides of the equation:

$$\mathcal{F}_\alpha \left[D_t^\beta u(t) \right] = \mathcal{F}_\alpha \left[f(t) \right] \tag{10.4}$$

Step 2: Substitution using Lemma 2.
Using **Lemma 2** for $n = 1$, and substituting the operational properties of the transform, we obtain:

$$\hat{u}_\alpha(u) \cdot \Phi(u, \alpha) = \hat{f}_\alpha(u) \tag{10.5}$$

Step 3: Definition of the Mittag-Leffler function.
By the definition of the Mittag-Leffler function $E_\beta(z)$, which is given by:

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)} \tag{10.6}$$

we have the following relationship for the solution:

$$u(t) = \int_{-\infty}^{\infty} K_\alpha(t, u) \hat{f}_\alpha(u) E_\beta(-\lambda t^\beta) \, du \tag{10.7}$$

Taking Inverse Fractional Fourier Transform:
By applying the operator $\mathcal{F}_{-\alpha}$ to both sides, we transition from the transform domain back to the time domain:

$$u(t) = \int_{-\infty}^{\infty} \hat{u}_\alpha(u) K_{-\alpha}(t, u) \, du \tag{10.8}$$

Substituting the algebraic expression obtained from Lemma 2, we have:

$$u(t) = \int_{-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(-\lambda t^\beta)^k}{\Gamma(\beta k + 1)} \right] \hat{f}_\alpha(u) K_{-\alpha}(t, u) \, du \tag{10.9}$$

Using the definition of the Mittag-Leffler function $E_\beta(z)$, the final solution is expressed as:

$$u(t) = \int_{-\infty}^{\infty} E_\beta(-\lambda t^\beta) \hat{f}_\alpha(u) K_{-\alpha}(t, u) \, du \tag{10.10}$$

References

[1] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons Inc., New York, 1993.
[2] R. Gorenflo and F. Mainardi, "Essentials of fractional calculus," *Preprint submitted to Maphysto Center*, January 28, 2000.
[3] K. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York–London, 1974.
[4] Haldun M. Ozaktas and M. Alper Kutay, "Introduction to the Fractional Fourier Transform and its Applications," *Advances in Imaging and Electron Physics*, Vol. 106, pp. 239–291, 1999.
[5] V. Namias, "The Fractional Order Fourier Transform and its Application to Quantum Mechanics," *J. Inst. Math. Appl.*, 25, pp. 241–265, 1980.
[6] H. Pollard, "The completely monotonic character of the Mittag-Leffler function," *Bull. Amer. Math. Soc.*, 54, pp. 1115–1116, 1948.

[7] S. M. Jung, "Approximate Solution of a linear differential equation of third order," *B. Malays. Math. Sci. Soc.*, 35, pp. 1063–1073, 2012.
[8] Y. Wang, Z. Zhou, and L. Sun, "Hyers–Ulam stability of linear differential equations of the first order," *Appl. Math. Lett.*, 21, pp. 1024–1028, 2008.
[9] I. A. Rus, "Ulam stabilities of ordinary differential equations in Banach space," *Carpathian J. Math.*, 26, pp. 103–107, 2010.
[10] Q. H. Alqifiary and S. M. Jung, "Laplace transforms and generalized Hyers–Ulam stability of linear differential equations," *Electron. J. Differential Equations*, 2014(2014), pp. 1–11.
[11] M. Onitsuka, "Hyers–Ulam stability of first-order linear differential equations of Caratheodory type and its application," *Appl. Math. Lett.*, 90, pp. 61–68, 2019.
[12] K. Liu, M. Feckan, and D. O'Regan, "Hyers–Ulam stability and existence of solutions for differential equations with Caputo–Fabrizio fractional derivative," *Mathematics*, 7(2019), 333.
[13] J. Wang and Y. Zhou, "Mittag-Leffler–Ulam stabilities of fractional evolution equations," *Appl. Math. Lett.*, 25(2012), pp. 723–728.

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